The fractal nature of everyday space

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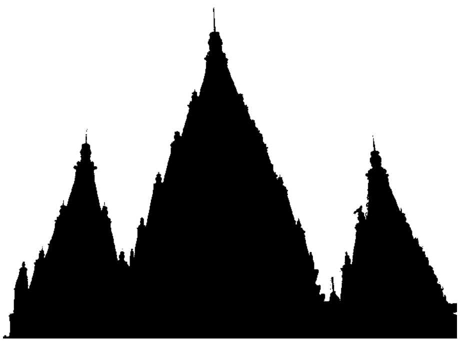
Abstract

Heaven is often portrayed as being fractal, not just in painting and architecture, but also in literature. Both C.S. Lewis and J.R.R. Tolkein depict a geometrically hyperbolic heaven that is fractal yet homely. Where does this idea come from? It is argued that picturesque everyday environments are actually slightly hyperbolic and images of heaven are an exaggeration of this quality.

The geometry of heaven

The first coloured illustration in Mandelbrot's *Fractal geometry of nature* is from an illuminated bible depicting Genesis 1:6 in which God separates the waters above from the waters below. The boundary so created is of course a fractal. This perhaps ought not be surprising because so many images associated in popular imagination with heaven, such as clouds, mountain ranges, music, titanic turbulence, hierarchies of angels and so forth are fractal. We note also that many religious buildings, especially those that to some degree try to represent heaven on earth, have a fractal aspect, and this is to be seen in buildings from many faiths. Gothic cathedrals resemble colossal sponges and repeat their motif of a pointed arch over perhaps seven orders of size, many classical buildings show an ordered repetition of spaces at different scales, Bramante's 1506 plan for St Peter's being a good example, but even these seem simple compared with those Hindu temples whose prodigious repetition and depth of order make them perhaps the most complicated buildings ever built. Figure 1 shows the obviously fractal silhouette of the temple at Prambanan. In a similar way the geometrical intricacies of Moorish architecture work at both large and small scales. The fantastic dream world architecture of the Alhambra Palace derives its power in part from its detail, everywhere you look, there is something curious to hold the eye. The Moorish style of fractal decoration of the door in figure 2 is by the remarkable English art and crafts architect Edgar Wood (1861 -1935) from his own house in Hale, Manchester (1914-16). The interior doors were painted by Wood himself and show the influence of his Moroccan sketching tours. It appears that a touch of mystery and grace is added to architecture by playing games with scale and repetition; heaven it seems is universally thought of as fractal.



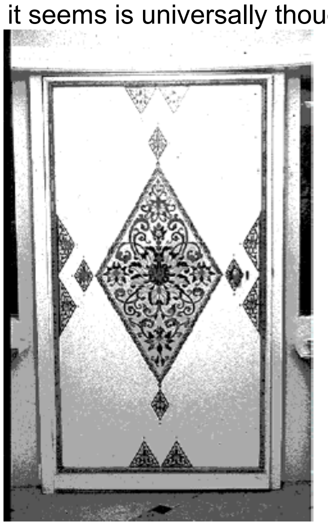


Figure 1; Temple at Prambanan, Indonesia.

Figure 2; Moorish door by Edgar Wood, c. 1916

A connection between heaven and fractals is also to be found in literature. Two books at least contain descriptions of worlds with a strong fractal nature, interestingly both are trying to portray heaven. They are *The Last Battle* by C.S. Lewis and *Tree and Leaf* by J.R.R. Tolkein. The first quotation below is taken from Tolkein who was writing in 1947. In this passage which comes close to the end of his story his hero, a failed painter called Niggle, without realising it at first, walks into heaven. It is, one might say, the ultimate happy ending.

... Niggle turned towards the Forest .... As he walked away, he discovered an odd thing: the Forest, of course, was a distant Forest, yet he could approach it even enter it, without its losing that particular charm. He had never before been able to walk into the distance without turning it into mere surroundings. It really added a considerable attraction to walking in the country, because, as you walked, new distances opened out: so that you now had double, treble, and quadruple distances, doubly, trebly, and quadruply enchanting. You could go on and on, and have a whole country in a garden, or in a picture (if you preferred to call it that). You could go on and on, but not perhaps for ever. There were Mountains in the background. They did get nearer, very slowly.

Tree and Leaf by J.R.R. Tolkein

The second quotation is from the last chapter of the last Narnia book written in 1956 by C.S. Lewis. The seven Narnia stories are a series of inter linked fantasy tales for children which form a religious allegory. At the end the children go to heaven.

'I see,' she said at last, thoughtfully. 'I see now. This garden is like the Stable. It is far bigger inside than it was outside.'

'Of course, Daughter of Eve,' said the Faun. 'The farther up and the farther in you go, the bigger everything gets. The inside is larger than the outside.'

Lucy looked hard at the garden and saw that it was not really a garden at all but a whole world, with its own rivers and woods and sea and mountains. But they were not strange: she knew them all.

'I see she said. 'This is still Narnia, and, more real and more beautiful than the Narnia down below, just as it was more real and more beautiful than the Narnia outside the Stable door! I see... world within world, Narnia within Narnia...'

'Yes,' said Mr Tumnus, 'like an onion: except that as you go in, each circle is larger than the last.'

And Lucy looked this way and that and soon found that a new and beautiful thing had happened to her. Whatever she looked at, however far away it might be, once she had fixed her eyes steadily on it, became quite clear and close as if she were looking through a telescope.

The Last Battle by C.S. Lewis

In both these happy endings space in heaven repeats itself inwardly in a way that mirrors the inner repetition of familiar fractals. Absolute size seems to have lost its meaning, Lewis's vision of heaven in particular is almost scale free. The paradoxical behaviour of near and yet not near reminds us of some of Mandelbrot's expectation paradoxes. (Mandelbrot, 1987)

How pleasing that the ending of the first story Tolkein wrote and the ending of the last children's story Lewis wrote should be so similar. Of course Tolkein and Lewis were close friends, intellectual collaborators almost, so the two stories have more in common than might be realised at first sight. Humphrey Carpenter's book *The Inklings* describes how they met with like minded friends in the Eagle and Child pub in Oxford where they read to each other from their work. It is therefore likely that they shared this idea of what heaven might be like. In both stories the characters are rewarded not for the large events things in their lives but for good deeds that seemed at the time to be unimportant. For both authors a supernatural hand links the large and small events in the plot. There is a sense in both that it is in the small events of life that true significance is to be found, a theme Tolkein developed in his Middle Earth stories where the really important work of saving the world is left to its least important inhabitants.

Now that the matter has been raised one can ask what previously might have seemed a peculiar question; what is the geometry of spaces in fantasy literature? Is Middle Earth fractal? At the very least it has a strange topology; places are connected in odd ways, by tunnels, seeing eyes, strange paths, and by the sudden transportation by eagles. Worlds which are separate yet similar lie close to each other and only occasionally touch. One is reminded of the strange and sudden connections between locations in the Alice stories or in Gulliver's Travels, both works that play with scale. enlarging and shrinking characters to allow new dramatic situation to arise. In short, geometry is something which can be adjusted to suit the dramatic convenience of the author, and the use of a geometrically peculiar environment is one way that an author can produce eerie or fantastic effects. Geometry cannot be too peculiar however, or the story loses credibility, one thinks of Tolkein's maps which reassure us that it is all real. Furthermore both Lewis and Tolkein took trouble to give their fantasies the inner consistency of reality, a process Tolkein called sub-creation. It is well known to what extraordinary lengths Tolkein went in giving his Middle Earth a geography, history and languages, even in regard to matters which did not appear in his novels, it was almost as if the creation of Middle Earth was the main thing and the novels a by-product of that activity. But Tolkein's heaven is certainly peculiar. Before examining it more closely it ought to be pointed out that both stories end with a description of heaven and its unusual geometrical properties do not really play any part in either plot. So we may conclude that Tolkein's heaven is an attempt to portray what he actually thought heaven might be like. Is there a sense in which his heaven could be at all realistic? I hope to show the everyday environment may be to some degree fractal, and the more fractal it is the more it corresponds to our notion of picturesque and enchanting, and conversely if it is not fractal it may seem oppressive and alien. It is by fractals that Tolkein's model of heaven may be connected to reality.

Hyperbolic space

All the images associated with heaven listed at the start are exaggerations of things familiar on earth, could this also be true of Tolkein's inwardly expanding model of heaven? The space he describes seems to be more than three dimensional, as if it had an excess of space; in short Tolkein's heaven is hyperbolic. Although difficult to visualise, a hyperbolic universe is not impossible or self contradictory. In such a world the angles of a triangle would sometimes add up to less that 180˚ so whether or not a space is to any extent hyperbolic is a question that can be settled by experiment. This question was first put to the test by Gauss in 1816 when he surveyed the angles of a 100 km triangle whose vertices were on mountain peaks. His experiment, and others since, have confirmed that at the scale of the earth space is flat not hyperbolic or elliptical. Despite this we can at least try to imagine what a hyperbolic world might be like. The usual way is to start with a hyperbolic surface, this being a thing that can be made and handled, and then imagine that a three dimensional hyperbolic world lies in the same relation to it as a cube does to a square. Hyperbolic surfaces look like seaweed, they cannot be ironed flat without folding, there is an excess of material causing it to crumple. Surfaces of this nature, sometimes called Thurston surfaces, can be knitted or crocheted or made out of paper by joining equilateral triangles seven rather than six at a point. The results are floppy and resemble some of the most fashionable models of up to date non-standard architecture.

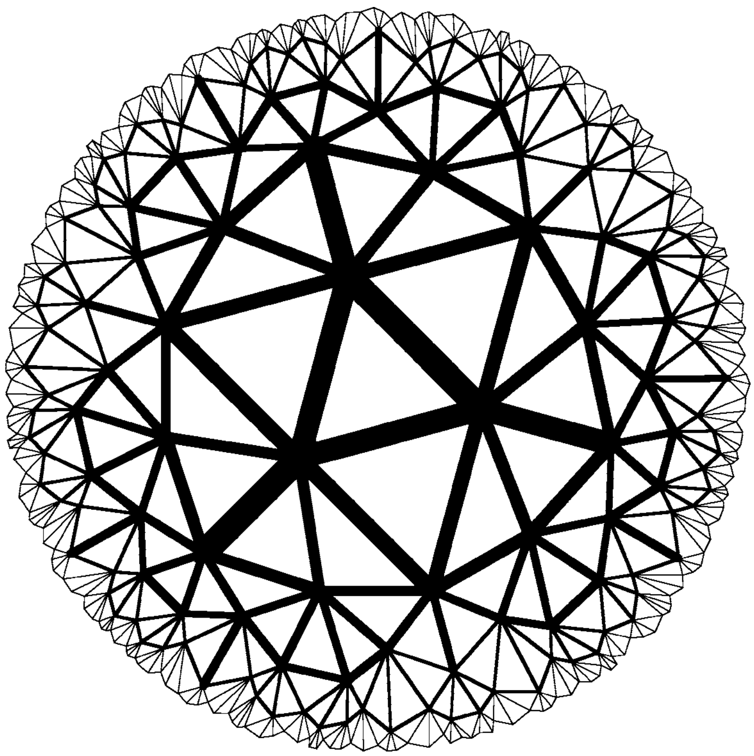


Figure 3; Tiling of triangles on a Thurston surface.

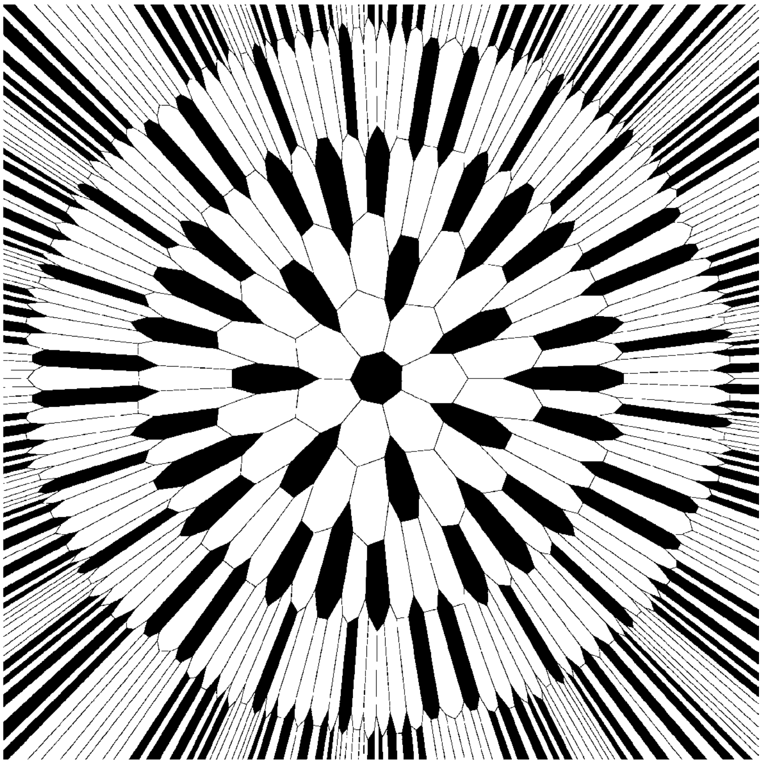


Figure 4; Fractal tiling of 7-gons.

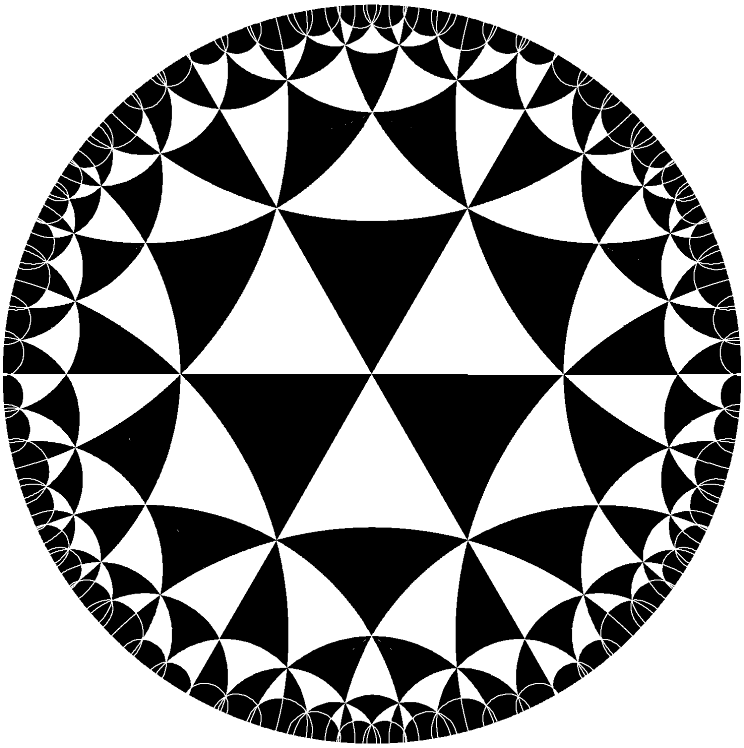


Figure 5; Poincaré disc model of hyperbolic space.

Hyperbolic space is naturally fractal. Figure 3 shows an attempt to represent a Thurston surface by means of a tiling of triangles meeting seven at each vertex. In a hyperbolic universe it would be possible to draw this pattern on flat paper keeping all the triangles the same size but in our flat Euclidian world in order to keep the drawing on the page the triangles must get smaller and the resulting pattern becomes a fractal tree. If the centres of adjacent triangles are joined by lines to construct the dual of figure 3 the resulting pattern is the fractal tiling of 7-gons shown in figure 4. Rather than shrink the cells uniformly the cells shrink only in their width remaining roughly all the same length. In a hyperbolic universe it would by possible to draw this explosive pattern with equal 7-gons. In both these examples impossible objects which exist properly only in hyperbolic space are represented by means of fractals.This trick works by shrinking space as one moves outwards. This brings us naturally to the most common way to visualise hyperbolic space; the Poincaré model. An infinite plane is mapped on to a disc in such a way that straight lines are represented by circles, as one moves from the centre to the perimeter distances are compressed so the edge of the disc represents points at an infinite distance from the middle. The image in figure 3 is a a tiling of equilateral triangles, sometimes they meet six at a point, sometimes eight. Moving in a hyperbolic universe is like moving on a plane where one shrinks as one moves away from the centre. Imagine walking from the centre going from triangle to triangle but shrinking as one travels so that all the triangles seem the same size, then the journey could last forever without ever reaching the edge, the circle contains an entire universe. This is the best image of hyperbolic space in our flat universe, we must imagine that we shrink as we move, and in so doing experience an excess of space that did not seem to be there before.

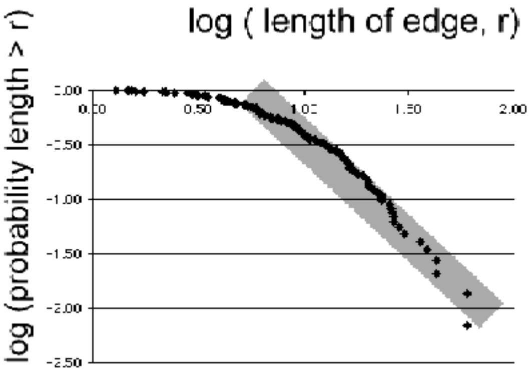


Figure 6; Distribution of edge lengths in figure 3.

Let us examine figure 3 to see how the shrinking is managed. The number of triangles in figure 3 grows prodigiously as one moves outwards. If starting with a seed triangle we build up patches of triangles in rings around the first then the number in the patch increases 1,4,13,41,77,125 for a hexagons meeting six at a point in the normal manner, but 1,4,16,61,181,499 for the hyperbolic space in figure 4. The rate of increase grows the larger the patch becomes. Another way to illustrate this growth is shown in figure 6 which is a Pareto plot showing the distribution of edge lengths in figure 3. Figure 3 was drawn by eye keeping the cells roughly similar, it is a sketch of a sort, yet the line is still fairly straight. The y axis shows log (P) where P is the probability that an edge in figure 3 has a length greater than r, the x axis shows log (r). The long edges are on the right, short ones on the left. It is the straight part of the graph, shaded grey,which is of interest, its slope tails off to the left because there are not enough short edges, this is because the pattern has been stopped too soon, if it was extended by drawing more smaller cells the line would be straight for longer. The graph tells us that if there are N edges of at least length r, then, roughly speaking.

log (N) = -2.0 log (r) + constant. (1)

This formula gives us some idea of how quickly excess space accumulates in the hyperbolic space illustrated in Figure 3. It is the form of the relationship which is significant here, by drawing the tiling slightly different, for example by changing the proportions of the outer cells the gradient of the line could be altered. If, for example, the sides of the outer cells were made shorter so compressing the pattern the gradient could be made less steep. This would correspond to a space which was even more exaggeratedly hyperbolic. If all the tiles had the same shape, as in an ordinary tiling of hexagons the line would be vertical. This formula and these diagrams are the model of hyperbolic space which correspond to the growth of space in Tolkein's story and will now be applied to the real world.

The geometry of the everyday environment

By space I mean that part of the world which is not occupied by objects. Looked at this way space is the most fantastically complicated thing imaginable. How then can it be described? A good beginning is to reverse solid and void and to treat the nothing of space as if it were a mould from which the world is pulled. It is by this method that Italian architect Luigi Moretti produced his solid plaster models of great architectural interior spaces. The shapes he crafted were described using ordinary geometry, indeed it is only by surveying the original spaces using the repertoire of shapes available to Euclidian geometry that the models could be made in the first place. More direct are the life-size casts of interior spaces by artist Rachel Whiteread who tackled the problem in a simpler way since she did not require an intermediate description of the space to be later recreated in plaster. However making space solid, although it is in a sense a description of space, somehow does not get to the heart of the matter. Robin Evans makes the following comment on Moretti's work, " ... the word space is as much about intimation as it is about surveyable dimensions. An intimation cannot be made of plaster, " (Evans, 1995). As I understand this remark, Evans is saying that to read a space depends on how we choose to live in it, that our sensation of space is influenced by its potential for human occupation. Models of space as the background made solid treat space as an object which is interesting up to a point but does not reveal how it can accommodate different objects and activities. An example might make this point clearer; here Robert Graves describes what appears to be a change in the nature of space, it changes size according to what you are doing.

A Second Battalion officer, who revisited these Laventie trenches after the war ended, told me the other day of the ridiculously small area of No Man's Land compared with its seeming immensity on the long, painful journeys that he had made over it. "It was like the size of a hollow in one's tooth compared with how it feels to the tongue."

Robert Graves, Goodbye to all that,

No-man's land in daytime peace and at night in war are quite different sizes. Such a change in size indicates that fractal might be at work. To study space which might be fractal one must first choose an activity and then see how the space can accommodate it at different scales. This is equivalent to the box counting method of analysing fractals (for example see Addison p.37). The activity could mean things as different as crawling about in fear of one's life or seeing how many boxes can be fitted in it. If the space is fractal then the size of the space will depend on the scale of the activity.

Ongoing experiments at the University of Manchester following this approach indicate that indeed our everyday space is probably slightly hyperbolic. These experiments involve measuring and analysing space as if it were a fresh and unusual object. Some experiments have investigated the capacity of the environment to house particular activities such as hiding places for children, others investigated spaces to place particular objects, in particular, parking cars. The fractal nature of the environment is revealed by scaling relationships between the number of objects which can be accommodated and their size. For example in the hide and seek experiment, (Crompton 2001) it was found that in an ordinary house the relationship between the number of hiding places, N, and the height of the child, r, was, roughly,

$$\text{Log}(N) = -2.4 \log(r) + \text{constant} \qquad (2)$$

This is equation should be compared to equation 1. What this means is that figure 3 provides a good model of the game hide and seek. If we say that choosing a hiding place is equivalent to choosing a line longer than a given value, then the number of lines to choose matches the number of hiding places in the house for a person of a particular height. Small people have a substantial advantage, they can find a lot more places to hide, for them space is bigger. If our model of hyperbolic space is that space expand as we get smaller as in the Poincaré model then the success of small children at hide and seek gives an active illustration of hyperbolic space. Other Manchester experiments, some immensely tedious involving measuring thousands of objects, have produced similar relationships for parking cars, housing furniture and household objects, places to read a book, as well as for fitting imaginary boxes into houses. All of them have produced scaling relationships similar to equations (1) and (2). This work is at present being prepared for publication.

Conclusion; Picturesque Space and Heaven

In certain environments there is an excess of small spaces which may be interpreted resembling a model of hyperbolic space and this is a quality associated with heaven. Does this happen everywhere or is the hyperbolic space more exaggerated in some places than in other? We can straight away point to places where it does not exist. These are be places which privilege a particular size, places such as car parks, prisons, modular buildings, large offices with arrays of equal desks, expanses of uniform curtain walling and so on. These are commonly thought of as oppressive places, perhaps one might say, elliptical. These are spaces which seem to fill up quickly where we are conscious of the presence of others, where we feel exposed. In short they are bad places to play hide and seek. The opposite, hyperbolic, sort of environment is found in complicated picturesque places, and this provides the clue to the relationship between heaven and hyperbolic space.

Picturesque design is likely to produce spaces that are fractal. Why this might be so has been outlined in a previous paper ‘Fractals and Picturesque composition.’ (Crompton 2002). The reasons broadly speaking are that rules for picturesque design are based on imitating nature and nature is already fractal. Here is one example from many others which could be given; artists in the nineteenth century and since who shared the romantic values of John Ruskin might follow his rules of picturesque composition. This could involve them, for example, in arranging their material to obey his ‘law of continuity’ which would have you lay out objects in slowly varying series rather than all equal and evenly spaced. Examples of such series include arches seen in perspective, banks of clouds, waves on a beach, the receding flanks of a valley and so on, (Ruskin 1857). Irregular series of slightly varying things were much admired by Ruskin The picturesque quality seen in a row of old posts, or a row of books is easily produced if their sizes vary according to a relationship like (1) or (2). If a random sample of objects is chosen from a set whose values vary hyperbolically then they will possess this desirable quality of irregularity This is because if the sizes of objects varies hyperbolically they will not appear to cluster around a mean value, there is always a small chance that outsize items will occur. But they are not disordered either, parts of the series when rescaled will resemble the whole. Such a collection will be a mixture of order and disorder; almost regular but not quite. Other examples of artistically pleasing hyperbolic relations which lie between order and chaos are to be found in Voss (1989).

Not only are fractal environments more natural, but they are more humane. The sort of advice given by Christopher Alexander and his followers (Alexander 1977) will tend to produce fractal spaces. A longer discussion of this point can be found in Crompton (2001). This is because Alexander concentrates on making spaces suited to the size of people doing homely things, and then makes all the spaces relate and connect to one another. This is the opposite approach to fitting activities to a universal grid, he delights in making lots of small spaces, not even forgetting that children need places suited to their own size and games. He concerns himself not only with buildings on their own but also with how they relate to one another right up to the scale of the town. This involves designing spaces over a range of scales and making them all join to one another. Although he is not overtly picturesque many of his exemplary illustrations are of picturesque scenes and it is not hard to see that if followed his rules for design would tend to produce complicated small scaled picturesque buildings set in public spaces with a range of scales, although not too big.

Why is heaven hyperbolic? Because it is an elaboration of picturesque places we enjoy on earth, and picturesque places are fractal and therefore already slightly hyperbolic. These are good places to live and good place to tell stories, here art and science cross. In a nutshell Hell may be elliptical but Heaven is hyperbolic.

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